Analytical solitary wave solutions for nonlinear Schrödinger equations with denominating saturating nonlinearity



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# Analytical solitary wave solutions for nonlinear Schrödinger equations with denominating saturating nonlinearity

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**Abstract.** Solitary wave solutions are determined analytically for two forms of nonlinear Schrödinger equations with saturation effects in the denominators of their nonlinearity terms. Pertinent parameters, being explicitly stated, make the results applicable to soliton phenomena.

## 1. Introduction

Waves which propagate with constant velocity and, in which dispersion (or diffraction) and nonlinearity harmonize to prevent dissipation are conventionally known as solitary waves. They occur both experimentally and theoretically [1] in many physical problems and applications of plasma physics [2, 3, 9], nonlinear surface waves [4, 5], pulse propagation in semiconductor doped glass fibres [6–8], and solids. Particularly for device modelling applications where saturation effects may become significant, the following forms of nonlinear Schrödinger equations are the dynamic governing equations

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t}+b\frac{\partial^2\Psi}{\partial x^2}-a\Psi+\frac{c|\Psi|^k\Psi}{1+\gamma|\Psi|^k}=0 \qquad bc>0 \tag{1}$$

$$\mathrm{i}\hbar\frac{\partial\Phi}{\partial t}+b\frac{\partial^2\Phi}{\partial x^2}-a\Phi+\frac{c|\Phi|^k\Phi}{1+\gamma|\Phi|^k}\pm\frac{d|\Phi|^{2k}\Phi}{1+\gamma|\Phi|^k}=0 \qquad bc>0 \quad d>0 \tag{2}$$

where $k=2n$, for $n=1,2,3,4,5,\ldots$.

Even for $k=2$, to our knowledge, only numerical solitary wave solutions of (1) and (2) are available. We present the fundamental analytical solutions in this paper. We have utilized the method of [1]. However, for the reason that collision between two solitary waves would, subject to some physical conditions, be either elastic or inelastic for which both earlier and recent research reports exist in the literature, one may sometimes refer to a solitary wave as a soliton. Additionally, analytical solutions (which may not be exact) of (1) and (2) are required for global understanding of relations between soliton parameters.

## 2. The analytical method

Saturation effects, physically characterized by $\gamma$ in (1) and (2), mainly delimit the magnitude of the maximum amplitude, which, in turn, sustains the practical limit of the soliton length (cf the cubic nonlinear Schrödinger equation, (CNLSE), [1], but note that the cubic quintic

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nonlinear Schrödinger equation, (CQNLSE), [1], is saturable). This suggests looking for the fundamental solution of (1) in the form

$$\Psi(x,t) = A_0\,\mathrm{e}^{\mathrm{i}(\alpha x-\alpha_0-\omega t)}\mathrm{sech}^{2/k}\frac{x-x_0-vt}{T_s} \tag{3}$$

so that the parameters $\alpha$, $\omega$, $A_0$, and $T_s$ have relating expressions to be determined. The initial conditions determine the phase constant, $\alpha_0$; $x_0$ has the meaning of soliton centre, and the soliton velocity, $v$, can be treated as a free parameter [1].

Substituting (3) into (1), yields

$$\begin{aligned}\left(\bar{h}\omega-\alpha^2 b-a+\frac{4b}{k^2T_s^2}\right)+\Bigg[&(\bar{h}\omega-\alpha^2b-a)\gamma A_0^k+cA_0^k\\ &-\frac{4b}{k^2T_s^2}+\frac{4b\gamma A_0^k}{k^2T_s^2}-\frac{2b}{kT_s^2}\left(\frac{2}{k}+1\right)\gamma A_0^k\mathrm{sech}^2\Lambda\Bigg]\mathrm{sech}^2\Lambda\\ &+\frac{2\mathrm{i}}{kT_s}(\bar{h}v-2b\alpha)(1+\gamma A_0^k\mathrm{sech}^2\Lambda)\tanh\Lambda\equiv 0\end{aligned} \tag{4}$$

where

$$\Lambda = \frac{x-x_0-vt}{T_s}. \tag{4$a$}$$

As indicated, equation (4) consists of three terms at the left-hand side, (LHS), such that every term is identically equal to zero. From the first term one obtains

$$\omega\bar{h} = \alpha^2 b + a - \frac{4b}{(kT_s)^2}. \tag{5}$$

The third term yields

$$\alpha = \frac{\bar{h}v}{2b} \tag{6}$$

and the second term leads to

$$A_0(x,t) = \left\{\frac{2b(2+k)}{ck^2T_s^2-2b(2+k)\gamma\,\mathrm{sech}^2\Lambda}\right\}^{1/k}. \tag{7}$$

At the centre of a solitary wave, the condition is such that

$$\Lambda = 0 \Rightarrow \mathrm{sech}\Lambda = 1 \tag{7$a$}$$

which applies to (7) to yield

$$A_0 = \left\{\frac{2b(2+k)}{ck^2T_s^2-2b(2+k)\gamma}\right\}^{1/k}. \tag{8}$$

A solitary wave solution of (1) is, therefore, given by

$$\Psi(x,t) = \left\{\frac{2b(2+k)}{ck^2T_s^2-2b(2+k)\gamma}\right\}^{1/k}\mathrm{e}^{\mathrm{i}(\alpha x-\alpha_0-\omega t)}\mathrm{sech}^{2/k}\frac{x-x_0-vt}{T_s}. \tag{9}$$

If the condition implied in (7$a$) is not effected by using (7) in (3), one gets another solitary wave solution that takes the following form

$$\Psi(x,t) = \mathrm{e}^{\mathrm{i}(\alpha x-\alpha_0-\omega t)}\left\{\frac{2b(2+k)}{ck^2T_s^2\cosh^2((x-x_0-vt)/T_s)-2b(2+k)\gamma}\right\}^{1/k}. \tag{10}$$

Equations (9) and (10) have the same magnitude of input amplitude given by equation (8) and soliton length, $T_s$, but, they differ in integral contents implying a difference in their input intensities. A significant finding, in some studies of equation (1) to be reported in a later paper, is that for $k = 2$ the stationary solitary wave profiles obtained from equation (9) agree with the well known numerically obtained profiles of [7].

Now, from the usual normalization of the propagating field, $\Psi$, in the form

$$I_0 = \int_{-\infty}^{\infty} |\Psi|^2 \, \mathrm{d}x \tag{11}$$

and using (9) in (11), the soliton width, $T_s$, can be determined from

$$I_0 = 2^{4/(k-1)} \frac{\Gamma^2(2/k)}{\Gamma(4/k)} T_s \left\{ \frac{2b(2+k)}{ck^2T_s^2 - 2b(2+k)\gamma} \right\}^{2/k} \tag{12}$$

where $\Gamma(\xi)$ is the gamma function, and $I_0$ has the meaning of conserved number of particles. We observe that if $\gamma = 0$, in equation (8), the expression for the maximum amplitude becomes

$$A_0 = \left[ \frac{2(k+2)}{k^2T_s^2} \frac{b}{c} \right]^{1/k} \tag{13}$$

which agrees with equation (5) of [1].

The parameter $c$, would normally be independent of $\gamma$ as in a few earlier works [6, 7, 10], and also by inference from actual experimental results [11]. However, if $c \propto \gamma$, we have a different form of denominating saturating nonlinearity such as that occurring in nonlinear plasma physics [9]. We thus allude to the physical significance that $c$, in its latter case, points to the agreement between denominating saturating nonlinearity and the exponential saturating nonlinearity [6, 7]. By agreement one refers to the common features: (i) if $\gamma = 0$, the medium is linear, and (ii) if $\gamma = \infty$, the medium is Kerr-like. The parameter $c$, in its former case, contrasts with those features, i.e. for $\gamma = 0$ or $\infty$, the medium is Kerr-like or linear.

Now we consider equation (2). For solitary waves, the following conditions are presently verifiable, (i) $b > 0$, $c > 0$ and $+d$; (ii) $b < 0$, $c < 0$, and $-d$. Confusion should not arise if one states results for which condition (i) applies, as follows. Before proceeding however, a peculiar physical application of equation (2) is recapitulated.

The equation is restricted more to the evolution of solitary waves in the phenomena of bistability, because nonlinearity takes the form reported by Gibbs *et al* [12], wherein $k = 2$. Also, it is the dynamic equation for soliton pulse propagation in double-doped semiconductor optical fibres [8]. In a broader analysis, the equation subsumes the generalized equations that satisfy multistability criteria in the sense established by Kaplan [13, 16]. For clarity, we thus present the analytical solution in two cognate stages: (i) two-state solution and (ii) quintic-like solution.

In the case of the two-state solution, one looks for a fundamental solution of the form

$$\Phi(x,t) = A_\mu \, \mathrm{e}^{\mathrm{i}(\alpha x - \alpha_0 - \Omega_a t)} \mathrm{sech}^{2/k} \frac{x - x_0 - vt}{T_a} \tag{14}$$

where $\mu$ connotes the possibility of two amplitude values corresponding to one value of soliton width, $T_a$. Here, $\mu = 1$ and $\mu = 2$ will refer to normal and saturated amplitudes respectively. All other parameters are as before.

Substituting (14) into (2) one obtains the following expressions for $\alpha$, $\Omega_a$ and the amplitudes $A_1$ and $A_2$

$$\hbar\Omega_a = \alpha^2 b + a - \frac{4b}{(kT_a)^2} \tag{15}$$

$$A_1 = \left[ \frac{1}{T_a^2} \frac{2(k+2)}{k^2} \frac{b}{c} \right]^{1/k} \tag{16}$$

$$A_2 = \left[ \frac{1}{T_a^2} \frac{2(k+2)}{k^2} \frac{b}{d} \gamma \right]^{1/k} \tag{17}$$

where $\alpha$ is given by equation (6). An important result in equation (17) is that the saturated amplitude $A_2 = 0$ if $\gamma = \mathbf{0}$, i.e. the medium becomes Kerr-like exclusively adapted to CNLSE, [1].

For the quintic-like solution, the nonlinearity power, $2k$, in the last term of equation (2), provides further insight that another solution exists with a different set of propagation parameters, which is an implication of the main multistability criterion [13]. Thus, as in [1] one seeks a fundamental solution of the form

$$\Phi(x,t) = B\,\mathrm{e}^{\mathrm{i}(\alpha x - \alpha_0 - \Omega_b t)} \left(1 + g \cosh \frac{x - x_0 - vt}{T_b}\right)^{-1/k}. \tag{18}$$

Henceforth, the principal problem is to determine the parameter $g$.

Substitution of (18) into (2) yields the following results

$$\bar{h}\Omega_b = \alpha^2 b + a - \frac{b}{(kT_b)^2} \tag{19}$$

$$g = \operatorname{sech}\eta \tag{20}$$

$$B = \left[ \frac{I_0}{T_b \Gamma(2/k) P_{-1/2}^{(k-4)/2k}(\cosh\eta)} \right]^{1/2} \frac{(\tanh\eta)^{1/k}}{(2\pi \sinh\eta)^{1/4}} \tag{21}$$

$$T_b = \left[ \frac{I_0}{\Gamma(2/k)(2\pi\sinh\eta)^{1/2} P_{-1/2}^{(k-4)/2k}(\cosh\eta)} \left( \frac{k^2}{2+k} \frac{c}{b} \tanh\eta \right)^{2/k} \right]^{k/(k-4)} \tag{22}$$

where, $\alpha$ is given by (6), $P_{-1/2}^{\nu}(\xi)$ is the Legendre function, and the following equation determines $\eta$

$$\frac{b(2+k)}{2ck^2(k+1)} \left\{ \frac{3\gamma + k\gamma + (k+1)\gamma \operatorname{sech}^2\eta - 2d(2+k)}{\tanh^2\eta} \right\} = \left\{ \frac{1}{(2\pi\sinh\eta)^{1/2}\Gamma(2/k) P_{-1/2}^{(k-4)/2k}(\cosh\eta)} \left( \frac{k^2}{2+k} \frac{c}{b} \tanh\eta \right)^{2/k} \right\}^{2k/(k-4)} \tag{23}$$

where $I_0$ is the conserved particle number given by equation (11).

Some remarks can be made about the parameter $d$. Mathematically, equation (2) may be considered as a special case of (1) if $d$ does not depend on $\gamma$. If $d = 0$, as an illustration, one obtains equation (1). Thus, all results of equations (19) to (23) make up another solution of (1), although equation (17) ceases to be physically valid. However, the physics of a given problem may be such that $d$ is a function of $\gamma$. As a typical case, it could be shown that $d$ is linearly related to $\gamma$ in [8].

For most applications, equation (23) leads to a transcendental equation for which $\eta$ is the unknown, i.e. $b$, $c$, $d$, $k$, $\gamma$ and $I_0$ are known. Desire for accuracy would then require any suitable numerical scheme to solve the resultant equation. At best, the equation solves graphically.

One of the current research interests is in the application of (2) to soliton pulse propagation with semiconductor double-doped optical fibres [8] as waveguides. The applicable dimensionless form of (2) can be shown to have typical parameter values $\bar{h} = 1$,

$a = 0$, $b = 1/2$, $c = 1$ and $k = 2$. For this case, the solitary wave solution yields the following

$$\frac{12\eta^2}{I_0^2} = \gamma(8 - 3\tanh^2\eta) - 8d \tag{24}$$

$$\Phi(x,t) = \left(\frac{I_0}{2T_b}\frac{\sinh\eta}{\eta}\right)^{1/2} \mathrm{e}^{\mathrm{i}(\alpha x - \alpha_0 - \Omega_b t)}\left(\cosh\eta + \cosh\frac{x - x_0 - vt}{T_b}\right)^{-1/2} \tag{25}$$

$$\alpha = v \qquad \Omega_b = \frac{1}{2}\left(v^2 - \frac{1}{4T_b^2}\right) \tag{26}$$

$$B = \left(\frac{I_0}{2T_b}\frac{\tanh\eta}{\eta}\right)^{1/2} \tag{27}$$

$$T_b = \frac{\eta}{I_0\tanh\eta} \tag{28}$$

where typical values of $\gamma$ and $d$ are available (see [8]) and typical values of $I_0$ are also obtainable from relevant literature (see [11]). Equations (17) and (27) are observed to be related.

## 3. Lagrangian formulation

It is simple to show that equations (1) and (2) can be expressed as variational problems corresponding to Lagrangians of the form

$$L_1 = \mathrm{i}\frac{\hbar}{2}\left(\Psi\frac{\partial\Psi^*}{\partial t} - \Psi^*\frac{\partial\Psi}{\partial t}\right) + \left(a - \frac{c}{\gamma}\right)|\Psi|^2 + b\left|\frac{\partial\Psi}{\partial x}\right|^2 + \frac{c}{\gamma}I_{L1} \tag{29}$$

$$L_2 = \mathrm{i}\frac{\hbar}{2}\left(\Phi\frac{\partial\Phi^*}{\partial t} - \Phi^*\frac{\partial\Phi}{\partial t}\right) + \left(a - \frac{c}{\gamma} \pm \frac{d}{\gamma^2}\right)|\Phi|^2 + b\left|\frac{\partial\Phi}{\partial x}\right|^2 \mp \frac{2d}{(2+k)\gamma}|\Phi|^{k+2} + \frac{1}{\gamma}\left(c \mp \frac{d}{\gamma}\right)I_{L2} \tag{30}$$

where $I_{L1}$ and $I_{L2}$ have the same form expressible as

$$I_{L1} = \frac{1}{\gamma^{1/n}}\int_0^\tau \frac{dy}{1+y^n} \tag{31}$$

$$\tau = \gamma^{1/2}|\Psi|^2 \qquad n = k/2 \tag{31$a$}$$

and, similarly for $I_{L2}$, $\tau = \gamma^{1/2}|\Phi|^2$. For every value of $n$ (i.e. $k$), $I_{L1}$ or $I_{L2}$ has a closed form [15]. Equations (29) and (30) can be used to study stability analysis [14].

In the case of $k = 2$ of which applications and physical problems are ubiquitous, the appropriate Lagrangians are

$$L_1 = \mathrm{i}\frac{\hbar}{2}\left(\Psi\frac{\partial\Psi^*}{\partial t} - \Psi^*\frac{\partial\Psi}{\partial t}\right) + \left(a - \frac{c}{\gamma}\right)|\Psi|^2 + b\left|\frac{\partial\Psi}{\partial x}\right|^2 + \frac{c}{\gamma}\ln(1+\gamma|\Psi|^2) \tag{32}$$

$$L_2 = \mathrm{i}\frac{\hbar}{2}\left(\Phi\frac{\partial\Phi^*}{\partial t} - \Phi^*\frac{\partial\Phi}{\partial t}\right) + \left(a - \frac{c}{\gamma} + \frac{d}{\gamma^2}\right)|\Phi|^2 + b\left|\frac{\partial\Phi}{\partial x}\right|^2 \mp \frac{d}{2}|\Phi|^4 + \frac{1}{\gamma}\left(c \mp \frac{d}{\gamma}\right)\ln(1+\gamma|\Phi|^2) \tag{33}$$

where an asterisk denotes a complex conjugate in equations (29), (30), (32) and (33). For pulse propagation in optical fibres, a variational approach [17] is possible.

For phenomena of $k \geqslant 4$ (i.e. $n \geqslant 2$), spikons [14] would be cited. If $n = 2$ or 3, corresponding expressions for $I_{L1}$ ($I_{L2}$) are as intermediately easy to obtain as for the case $n = 1$, [15]. However, from [15], we give below the corresponding expressions for $n \geqslant 4$

$$I_{L1}(k \geqslant 8) = \frac{1}{\gamma^{1/n}}\left\{-\frac{2}{n}\sum_{q=0}^{(n/2)-1} P_q \cos\frac{2q+1}{n}\pi + \frac{2}{n}\sum_{q=0}^{(n/2)-1} Q_q \sin\frac{2q+1}{n}\pi\right\} \tag{34}$$

for $n \geqslant 2r$; $r = 2, 3, 4, 5, \ldots,$ i.e. even values of $n$; and

$$I_{L1}(k \geqslant 8) = \frac{1}{\gamma^{1/n}}\left\{\frac{1}{n}\ln(1+\tau) - \frac{2}{n}\sum_{q=0}^{(n-3)/2} P_q \cos\frac{2q+1}{n}\pi + \frac{2}{n}\sum_{q=0}^{(n-3)/2} Q_q \sin\frac{2q+1}{n}\pi\right\} \tag{35}$$

for $n \geqslant (2r+1)$; $r = 2, 3, 4, 5, \ldots,$ i.e. odd values of $n$; where $P_q$ and $Q_q$ are defined thus

$$P_q = \frac{1}{2}\ln\left(\tau^2 - 2\tau\cos\frac{2q+1}{n}\pi + 1\right) \tag{36}$$

$$Q_q = \tan^{-1}\left\{\frac{\tau\sin((2q+1)/n)\pi}{1-\tau\cos((2q+1)/n)\pi}\right\} \tag{37}$$

with $\tau$ given by equation (31$a$).

Either from the Lagrangians or directly from (1) and (2), one can work out the first three conserved quantities. The first integral is of the form given by equation (11); the second, which corresponds to the momentum, is of the form

$$I_1 = \mathrm{i}\hbar\int_{-\infty}^{\infty}\left(\Phi\frac{\partial\Phi^*}{\partial x} - \Phi^*\frac{\partial\Phi}{\partial x}\right)\mathrm{d}x. \tag{38}$$

The third integral corresponds to the conserved energy which thus depends on $k$. If $k = 2$, the third integral of equation (2) is

$$I_2 = \int_{-\infty}^{\infty}\left\{2b\left|\frac{\partial\Phi}{\partial x}\right|^2 + \left(a - \frac{c}{\gamma} \pm \frac{d}{\gamma^2}\right)|\Phi|^2 \mp \frac{d}{2\gamma}|\Phi|^4 + \frac{1}{\gamma^2}\left(c \mp \frac{d}{\gamma}\right)\ln(1+\gamma|\Phi|^2)\right\}\mathrm{d}x. \tag{39}$$

To obtain the third integral of (1) one puts $d = 0$ in (39), and additionally, $d$ may not depend on $\gamma$, but, one should also realize a theoretical implication that another set of propagation parameters are then deducible. By inference, as noted earlier, the results of [8] imply that for $d = 0$, the saturation parameter $\gamma = 0$, i.e. $d$, depends on $\gamma$.

If $k \geqslant 2$, the third integral is not explicitly obtainable. The expression will involve series, thus requiring $\gamma|\Phi|^k < 1$; this implies that approximation is inevitable.

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## References

[1] Pushkarov Kh I and Pushkarov D I 1980 *Rep. Math. Phys.* **17** 37–40 and references therein
[2] Sheerin J P and Ong R S B 1977 *Phys. Lett.* **63A** 279–80
[3] Shukla P K, Yu M Y and Spatschek K H 1977 *Phys. Lett.* **62A** 332–4
[4] Mihalache K and Mazilu D 1986 *Solid State Commun.* **60** 397–9
[5] Akhmediv N N, Nabiev R F and Popov Yu M 1981 *Solid State Commun.* **66** 981–5

[6] Kumar A, Kurz T and Lauterborn W 1996 *Phys. Rev.* E **53** 1166–71
[7] Gatz S and Hermann J 1991 *J. Opt. Soc. Am.* B **8** 2296–302
[8] Gatz S and Hermann J 1992 *Opt. Lett.* **17** 484–6
[9] Anderson D and Bonnedal M 1979 *Phys. Fluids* **22** 105–9
[10] Miguel H J, Cavalcanti S B, Borges N M, Gouveia E A and Gouveia-Neto A S 1993 *Opt. Lett.* **18** 182–4
[11] Coutaz J-L and Kull M 1991 *J. Opt. Soc. Am.* B **8** 95–8
[12] Gibbs H M, McCall S L, Venkatesan T N C, Gossard A C, Passner A and Wiegmann W 1979 *Appl. Phys. Lett.* **35** 451–543
[13] Kaplan A E 1985 *Phys. Rev. Lett.* **55** 1291–4
[14] Laedke E W and Spatchek K H 1979 *Phys. Lett.* **74A** 205–7
Malaomed B and Weinetein M I 1996 *Phys. Lett.* **220A** 91–6
[15] Gradshteyn I S and Ryzhik I M 1979 *Table of Integrals, Series, and Products* (New York: Academic) p 63
[16] Enns R H, Rangnekar S S and Kaplan A E 1987 *Phys. Rev.* A **36** 1270–9
[17] Anderson D 1983 *Phys. Rev.* A **27** 3135–45